The importance of being discrete: on the (in-)accuracy of continuous approximations in auction theory

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Motivation

- Auction theory is ‘doubly continuous’
- But real world (and experimental) auctions are ‘doubly discrete’
- Does this matter?

Existing (theoretical) papers

- Discrete values and continuous bids: Maskin and Riley (1985), Riley (1989), Wang (1991), Che and Gale (2006) and Bergemann, Brooks and Morris (2017)
- Discrete bids and continuous values: Chwe (1989), Rothkopf and Harstad (1994), David et al. (2007), Cai, Wurman and Gong (2010) and Gonçalves and Ray (2017)
- Discrete bids and discrete values: Dekel and Wolinsky (2003), Robles and Shimoto (2012)

Existing (experimental) papers

- Experimenters are fond of testing continuous models with discrete experiments
- Usually, they ignore this issue
- Occasionally, it is addressed but not in a rigorous way (e.g. Noussaire and Silver, 2005)
- Exception: Goeree, Holt and Palfrey (2002)

Model

- $n \geq 2$ risk neutral bidders
- Private values are independently drawn from the set $X = \{0, \delta, 2\delta, ..., x\delta\}$ for some $\delta > 0$ and $x \in \mathbb{N}$
- In line with many auction experiments, bids must belong to X
- A (pure) strategy is a function $\beta_i : X \to X$
- Focus on pure-strategy equilibria in undominated strategies
- Study 3 'standard' auction formats
- Consider two tie-breaking rules; here will focus on case without ties

Main finding

Introducing arbitrarily small amounts of discreteness can rob auctions of pure strategy equilibria.

Second price auctions

Proposition

Bidding one's valuation is a weakly dominant strategy in the second-price auction. Furthermore, in any equilibrium in undominated strategies, all players set $\beta(v) = v$ or $\beta(v) = v + \delta$.

First-price and all-pay auctions: building blocks

Lemma (Normalisation)

The set of equilibria does not depend on δ.

First-price and all-pay auctions: building blocks

Lemma (No dominated strategies)

Suppose that a bidding function β is chosen in an equilibrium. Then $\beta(0) = 0$, $\beta(1) \leq 1$ and $\beta(v) \leq v - 1$ for all $v \geq 2$.

First-price and all-pay auctions: building blocks

Lemma (Monotonicity)

Bids are weakly increasing in valuations.

First-price and all-pay auctions: building blocks

Lemma (No jumps)

In any SE, if $\beta(v) = b$*, then* $\beta(v+1) \leq b+1$*.*

First-price and all-pay auctions: building blocks

Lemma (Uniqueness)

The SE (if it exists) is unique up to a choice of $\beta(1)$.

A distributional assumption

- These lemmas hold regardless of the distribution generating values
- From now on, however, assume uniform values

First price auctions

Proposition

In the first price auction with two bidders, the bidding functions $\beta(v) = \lfloor v/2 \rfloor$ and $\beta(v) = \lceil v/2 \rceil$ both constitute SE. Furthermore, they are the only SE.

Corollary

In any SE with two bidders, bidding functions, expected payments and expected revenues converge to their counterparts in the continuous auction as $\delta \to 0$.

First-price auctions

Proposition

Suppose that there are 3 or more bidders and that the maximum x satisfies

$$x > \frac{2^{\frac{1}{n-1}}}{2^{\frac{1}{n-1}} - 1}.$$

Then there are no SE in the first-price auction.

First-price auctions

- Condition on number of values seems reasonable (e.g. if $x = 100$, we require $n \leq 69$)
- There *is* an SE in the very high bidder case
- Non-existence also common with ties (e.g. $n = 2$, even maximum)
- Result contradicts assertions by several experimenters, e.g. Cox, Roberson and Smith (1982)

All-pay auctions

Proposition

Suppose that $x \geq 10$*. Then, regardless of the number of bidders, there are no SE in the all-pay auction.*

All-pay auctions

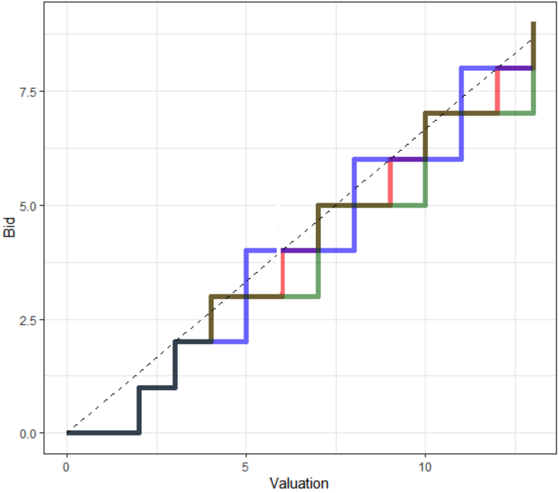
- Again, contradicts claims by experimenters (e.g. Noussaire and Silver, 2006)
- Also corrects their claim that discreteness only matters because it introduces a strictly positive probability of winning through a tie

Asymmetric equilibria

To investigate asymmetric equilibria, use a computational approach:

- Write down games in normal form (varying n, auction structure, tie-breaking rule)
- Use the Dekel-Fudenberg procedure to reduce the number of strategies
- Compute equilibria with Gambit

Asymmetric equilibria



Asymmetric equilibria

- However, in 6 out of 7 auction structures we consider, find games which lack PSNE
- Emphasises non-existence problem highlighted earlier

Playing with the discretisation

- Is it possible to preserve the predictions of continuous theory by discretising bids and values in a different way?
- More concretely, let $F_C : [\underline{v}, \bar{v}] \rightarrow [0,1]$ denote the continuous CDF and let β denote the SE of the (doubly) continuous auction
- Q: does there exist a finite set of possible bids B and finite set of possible values V such that $\beta(v) = \beta_C(v)$ for all $v \in V$?

Playing with the discretisation

Recipe:

1. Choose some $\delta > 0$ and construct a discrete distribution F_D over the set $\mathrm{V} = \{\underline{v} - \delta, \underline{v}, ..., \bar{v} - \delta\}$ using $F_D(v) = F_C(v + \delta)$ for any $v \in \mathrm{V}$
2. Let $\mathrm{B} = \{\beta_C(\underline{v}), \beta_C(\underline{v} + \delta), ..., \beta_C(\bar{v})\}$
3. No ties

Playing with the discretisation

Proposition

In the first-price and all-pay auctions, there is an SE in which each player bids $\beta_C(v)$ *for all* $v \in V$

Discussion

- As we have seen, introducing arbitrarily small amounts of discreteness can rob auctions of pure strategy equilibria
- In such cases, the only equilibria are in mixed strategies
- Does this matter?

Discussion

Implications for experimenters:

- Dangerous to unreflectively test continuous theory using discrete experiments
- Instead, could use our computational approach or discretise in a way that guarantees preservation of continuous predictions
- Or can calculate equilibria directly (easy in model without ties using our lemmas and their obvious generalisations)

Discussion

Implications for theorists:

- Lots of work done to establish existence of pure strategy equilibria (e.g. Maskin and Riley, 2000; Athey, 2001, Reny and Zamir, 2004)
- However, this finding is based on (inaccurate) continuous approximations

Discussion

Is it plausible that people randomise in auctions (with the equilibrium probabilities, and even though they have no strict incentive to do so?)

- 'Classical' justification of mixed strategies (von Neumann and Morgenstern, 1944) has even less force than usual in this context
- Empirically, we suspect that randomisation in auctions is rare
- Increases the cognitive requirements underlying BNE
- Undermines learning justifications?

Insofar as we find the idea of random bidding implausible, these findings push us in the direction of scepticism towards classical auction theory